Hydrodynamic Turbulence Has Infinitely Many Anomalous Dynamical Exponents

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On the basis of the Navier-Stokes equations we develop the statistical theory of many space-time correlation functions of velocity differences. Their time dependence is *not* scale invariant: n-order correlations functions exhibit $n-1$ distinct decorrelation times that are characterized by $n-1$ anomalous dynamical scaling exponents. We derive exact scaling relations that bridge all these dynamical exponents to the static anomalous exponents ζ_n of the standard structure functions.

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Experimental investigations of the statistical objects that characterize the small scale structure of turbulent flows are almost invariably based on a single point measurement of the velocity field as a function of time [1–4]. The Taylor "frozen turbulence" hypothesis is then used to surrogate time for space. The results of this type of measurements are "simultaneous" correlation of the velocity field itself or of velocity differences across a scale R (structure functions), or of velocity gradient fields like the dissipation field. Theoretical analysis which starts with the Navier-Stokes equations, on the other hand, states unequivocally that a closed form theory for the simultaneous many space-point correlation functions of velocity differences is not available. The perturbative theory of turbulence is very clear about this: attempting to derive equations for simultaneous correlation functions one finds integrals over time differences of space-time correlation and response functions [5]. There are no small parameters like a ratio of time scales (as in turbulent advection [6]) or a small interaction (like in weak turbulence [7]) that allow a reduction of such a theory to a closed scheme in terms of simultaneous objects only. The aim of this Letter is to initiate a *nonperturbative* analytic theory of n-order space-time correlation functions, and to find the characteristics of such a theory that can be related to objects that are known from standard experiments. It should be pointed out that in addition to the fundamental interest of such a theory it has important applications for the theory of scalar advection [8,9]; such a theory is beyond the scope of this Letter and is only mentioned as an additional motivation for the present analysis.

In recent theoretical work [10,11] about the same time statistics of turbulence it was made clear why it is extremely advantageous to consider many point correlation functions. The basic field under study is the difference of the Eulerian velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ across a scale $\boldsymbol{R}\equiv\boldsymbol{r}'-\boldsymbol{r}$: $\boldsymbol{w}(\boldsymbol{r},\boldsymbol{r}',t)\equiv\boldsymbol{u}(\boldsymbol{r}',t)-\boldsymbol{u}(\boldsymbol{r},t)$. The fundamental statistical quantities are the simultaneous "fully unfused" n-rank tensor correlation function of velocity differences:

$$\boldsymbol{F}_n(\boldsymbol{r}_1,\boldsymbol{r}'_1;\boldsymbol{r}_2,\boldsymbol{r}'_2;\dots;\boldsymbol{r}_n,\boldsymbol{r}'_n) \\ =\langle \boldsymbol{w}(\boldsymbol{r}_1,\boldsymbol{r}'_1,t)\boldsymbol{w}(\boldsymbol{r}_2,\boldsymbol{r}'_2,t)\dots\boldsymbol{w}(\boldsymbol{r}_n,\boldsymbol{r}'_n,t)\rangle \ , \tag{1}$$

where pointed brackets denote the cumulant part of the average over a (time-stationary) ensemble. In this quantity all the coordinates are distinct. The more usual structure function $S_n(R)$

$$S_n(R)=\langle |\ \boldsymbol{w}(\boldsymbol{r},\boldsymbol{r}',t)|^n\rangle \ , \quad \boldsymbol{R}\equiv\boldsymbol{r}'-\boldsymbol{r} \ , \tag{2}$$

is obtained by fusing all the coordinates $\boldsymbol{r}_i$ into one point $\boldsymbol{r}$, and all the coordinates $\boldsymbol{r}'_i$ into another point $\boldsymbol{r}+\boldsymbol{R}$. Obviously, in using the functions of many variables $\boldsymbol{F}_n$ instead of the one variable function $S_n(R)$ one is paying a heavy price. On the other hand this has an enormous advantage: when one develops the theory for $S_n(R)$ on the basis of the Navier-Stokes equations one encounters the notorious closure problem: knowing $S_n(R)$ requires information about S_{n+1} etc. It is well known that arbitrary closures of this hierarchy of equations are doomed, leading to predictions that are in contradiction with experiments. On the other hand no one succeeded to solve the hierarchy in its entirety. In contradistinction, the theory for the fully unfused $\boldsymbol{F}_n$ does not suffer from this problem: it was shown that there exist homogenous equations for $\boldsymbol{F}_n$ in terms of $\boldsymbol{F}_n$, without any hierarchic connections to higher or lower order correlation functions. This fact allows one to proceed [10,11] to derive a variety of exact bridge relations between the scaling exponents of gradient fields and the scaling exponents of the structure functions themselves, and to study the nature of the dissipative scales in turbulence, showing that in fact they are scaling functions with well defined scaling exponents.

In considering the decorrelation times of many "fully unfused" space-*time* correlation functions we need to make a choice of which velocity field we take as our fundamental field. The Eulerian velocity field won't do, simply because its decorrelation time is dominated by the sweeping of small scales by large scale flows. In [5] we showed that at least from the point of view of the perturbative theory one can get rid of the sweeping effect using the Belinicher-L'vov velocity fields whose decorrelation time is intrinsic to the scale of consideration. In terms of the Eulerian velocity ref. [12] defined the field $\boldsymbol{v}(\mathbf{r}_0,t_0|\boldsymbol{r},t)$ as

$$\boldsymbol{v}(\boldsymbol{r}_0,t_0|\boldsymbol{r},t)\equiv\boldsymbol{u}[\boldsymbol{r}+\boldsymbol{\rho}_{\rm L}(\boldsymbol{r}_0,t_0|t),t] \ , \tag{3}$$

$$\boldsymbol{\rho}_{\rm L}(\boldsymbol{r}_0,t_0|t)=\int_{t_0}^{t}\boldsymbol{u}[\boldsymbol{r}_0+\boldsymbol{\rho}_{\rm L}(\boldsymbol{r}_0,t_0|\tau),\tau] \ . \tag{4}$$

The observations of Belinicher and L'vov was that the variables $\boldsymbol{v}(\boldsymbol{r}_0, t_0|\boldsymbol{r}, t)$ satisfy a Navier-Stokes-like equation in the limit of incompressible fluid, and that their simultaneous correlators are identical to the simultaneous correlators of $\boldsymbol{u}(\boldsymbol{r}, t)$.

Introduce now a difference of two (simultaneous) BL-velocities at points $\boldsymbol{r}$ and $\boldsymbol{r}'$

$$\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0, t_0|\boldsymbol{r}, \boldsymbol{r}', t) \equiv \boldsymbol{v}(\boldsymbol{r}_0, t_0|\boldsymbol{r}, t) - \boldsymbol{v}(\boldsymbol{r}_0, t_0|\boldsymbol{r}', t) \ . \quad (5)$$

The equation of motion for $\boldsymbol{\mathcal{W}}$ can be calculated starting from the Navier-Stokes equation for the Eulerian field,

$$\Big[\frac{\partial}{\partial t} + \hat{\mathcal{L}}(\boldsymbol{r}, \boldsymbol{r}', t) - \nu(\nabla_r^2 + \nabla_r'^2)\Big]\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0, t_0|\boldsymbol{r}, \boldsymbol{r}', t) = 0. \quad (6)$$

We introduced an operator $\hat{\mathcal{L}}(\boldsymbol{r}_0, t_0|\boldsymbol{r}, \boldsymbol{r}', t)$

$$\begin{aligned}\hat{\mathcal{L}}(\boldsymbol{r}_0, t_0|\boldsymbol{r}, \boldsymbol{r}', t) \equiv & \overleftrightarrow{\boldsymbol{P}}\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0, t_0|\boldsymbol{r}, \boldsymbol{r}_0, t) \cdot \boldsymbol{\nabla}_r \\ & + \overleftrightarrow{\boldsymbol{P}}'\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0, t_0|\boldsymbol{r}', \boldsymbol{r}_0, t) \cdot \boldsymbol{\nabla}_r' \ , \end{aligned} \quad (7)$$

where $\overleftrightarrow{\boldsymbol{P}}$ is the usual transverse projection operator which is formally written as $\overleftrightarrow{\boldsymbol{P}} \equiv -\nabla^{-2}\boldsymbol{\nabla} \times \boldsymbol{\nabla}\times$. The application of $\overleftrightarrow{\boldsymbol{P}}$ to any given vector field $\boldsymbol{a}(\boldsymbol{r})$ is non local, and it has the form:

$$[\overleftrightarrow{\boldsymbol{P}}\boldsymbol{a}(\boldsymbol{r})]_\alpha = \int d\tilde{\boldsymbol{r}} P_{\alpha\beta}(\boldsymbol{r} - \tilde{\boldsymbol{r}}) a_\beta(\tilde{\boldsymbol{r}}). \quad (8)$$

The explicit form of the kernel can be found, for example, in [5]. In (7) $\overleftrightarrow{\boldsymbol{P}}$ and $\overleftrightarrow{\boldsymbol{P}}'$ are projection operators which act on fields that depend on the corresponding coordinates $\boldsymbol{r}$ and $\boldsymbol{r}'$. The equation of motion (6) form the basis of the following discussion of the time correlation functions.

To simplify the appearance of the fully unfused, multi-time correlation function of BL-velocity differences we choose the economic notation $\boldsymbol{\mathcal{W}}_j \equiv \boldsymbol{\mathcal{W}}(\boldsymbol{r}_0, t_0|\boldsymbol{r}_j\boldsymbol{r}'_j, t)$:

$$\boldsymbol{\mathcal{F}}_n(\boldsymbol{r}_0, t_0|\boldsymbol{r}_1\boldsymbol{r}'_1, t_1 \dots \boldsymbol{r}_n\boldsymbol{r}'_n, t_n) = \langle \boldsymbol{\mathcal{W}}_1 \dots \boldsymbol{\mathcal{W}}_n \rangle \ . \quad (9)$$

We begin the development with the simplest non- simultaneous case in which there are two different times in (9). Chose $t_i = t+\tau$ for every $i \le p$ and $t_i = t$ for every $i > p$. We will denote the correlation function with this choice of times as $\boldsymbol{\mathcal{F}}^{(p)}_{n,1}(\tau)$, omitting for brevity the rest of the arguments. Compute the time derivative of $\boldsymbol{\mathcal{F}}^{(p)}_{n,1}$ with respect to τ:

$$\frac{\partial \boldsymbol{\mathcal{F}}^{(p)}_{n,1}(\tau)}{d\tau} = \sum_{j=1}^{p} \langle \boldsymbol{\mathcal{W}}_1 \dots \frac{\partial \boldsymbol{\mathcal{W}}_j}{\partial t} \dots \boldsymbol{\mathcal{W}}_n \rangle \ , \quad (10)$$

Using the equation of motion (6) we find

$$\frac{\partial \boldsymbol{\mathcal{F}}^{(p)}_{n,1}(\tau)}{d\tau} + \boldsymbol{\mathcal{D}}^{(p)}_{n,1}(\tau) = \boldsymbol{\mathcal{J}}^{(p)}_{n,1}(\tau) \ , \quad (11)$$

$$\boldsymbol{\mathcal{D}}^{(p)}_{n,1}(\tau) = \sum_{j=1}^{p} \langle \boldsymbol{\mathcal{W}}_1 \dots \hat{\boldsymbol{\mathcal{L}}}_j \boldsymbol{\mathcal{W}}_j \dots \boldsymbol{\mathcal{W}}_n \rangle \ , \quad (12)$$

$$\boldsymbol{\mathcal{J}}^{(p)}_{n,1}(\tau) = \nu \sum_{j=1}^{p} (\nabla_j^2 + {\nabla'}_j^2) \langle \boldsymbol{\mathcal{W}}_1 \dots \boldsymbol{\mathcal{W}}_j \dots \boldsymbol{\mathcal{W}}_n \rangle \ , \quad (13)$$

with $\hat{\boldsymbol{\mathcal{L}}}_j \equiv \hat{\boldsymbol{\mathcal{L}}}(\boldsymbol{r}_0, t_0|\boldsymbol{r}_j, \boldsymbol{r}'_j, t)$. We remember that $\hat{\boldsymbol{\mathcal{L}}}_j\boldsymbol{\mathcal{W}}_j$ is a nonlocal object that is quadratic in BL-velocity differences, cf. Eq.(7). We reiterate that all the functions depend on $2n$ space coordinates that we do not display for notational economy.

To understand the role of the various terms in Eq.(11) we will make use of the analysis of a similar equation that was presented in [11]. In that case we considered the simultaneous object $\boldsymbol{F}_n = \boldsymbol{\mathcal{F}}^{(p)}_{n,1}(0)$ and computed, as above, its t-time derivative. Obviously, in the stationary ensemble this derivative vanishes. Instead of $\boldsymbol{\mathcal{D}}^{(p)}_{n,1}(\tau)$ and $\boldsymbol{\mathcal{J}}^{(p)}_{n,1}(\tau)$ we got $\boldsymbol{\mathcal{D}}^{(n)}_{n,1}(\tau)$ and $\boldsymbol{\mathcal{J}}^{(n)}_{n,1}(\tau)$ which differ from the present ones only in that the summation go up to n instead of p. The crucial observations of [11] are the following ones: (i) $\lim_{\nu\to o} \boldsymbol{\mathcal{J}}^{(n)}_{n,1}(\tau) = 0$. For fully unfused simultaneous correlation functions the viscous term in the balance equation disappears in the limit of vanishing viscosity. (ii) The integral in $\boldsymbol{\mathcal{D}}^{(n)}_{n,1}(\tau)$ which originates from the projection operator converges in the infra-red and the ultraviolet regimes. This means that every term in the sum over j that contributes to $\boldsymbol{\mathcal{D}}^{(n)}_{n,1}(\tau)$ can be estimated as $S_{n+1}(R)/R$ when all the separation $\boldsymbol{R}_j \equiv \boldsymbol{r}_j - \boldsymbol{r}'_j$ are of the same order R. When we take the full sum up to n there exist internal cancellations between all these terms, leading to the homogeneous equation $\boldsymbol{\mathcal{D}}^{(n)}_{n,1} = 0$.

In the present analysis the proof that the viscous term $\boldsymbol{\mathcal{J}}^{(p)}_{n,1}(\tau)$ is negligible when $\nu \to 0$ is an immediate consequence of the previous result. Since we are taking only partial sums on j, the internal cancellation in$\boldsymbol{\mathcal{D}}^{(p)}_{n,1}(\tau)$ disappears, and it has a finite limit when $\nu \to 0$. On the other hand $\boldsymbol{\mathcal{J}}^{(p)}_{n,1}(\tau)$ can only increase if we take $\tau = 0$. Thus again when $\nu \to 0$ we can neglect $\boldsymbol{\mathcal{J}}^{(p)}_{n,1}(\tau)$ in the fully unfused situation. Accordingly for $\nu \to 0$ we have

$$\partial \boldsymbol{\mathcal{F}}^{(p)}_{n,1}(\tau)/d\tau + \boldsymbol{\mathcal{D}}^{(p)}_{n,1}(\tau) = 0 \ . \quad (14)$$

The proof of convergence of the integrals in $\boldsymbol{\mathcal{D}}^{(p)}_{n,1}(\tau)$ follows from the previous results, since the time correlation functions are bounded from above by the simultaneous ones. Accordingly, when all the coordinates are fully unfused and the separations are all of the order of R,

$$\boldsymbol{\mathcal{D}}^{(p)}_{n,1}(\tau) \sim \boldsymbol{\mathcal{F}}^{(p+1)}_{n+1,1}(\tau)/R \ . \quad (15)$$

Introduce now the typical decorrelation time $\tau^{(p)}_{n,1}(R)$ that is associated with the one-time difference quantity $\boldsymbol{\mathcal{F}}^{(p)}_{n,1}(\tau)$ when all the separations are of the order of R:

$$\int_{-\infty}^{0} d\tau \boldsymbol{\mathcal{F}}^{(p)}_{n,1}(\tau) \equiv \tau^{(p)}_{n,1} \boldsymbol{\mathcal{F}}^{(p)}_{n,1}(0) \ . \quad (16)$$

Remember that the simultaneous correlation functions of BL-velocity differences (9) are identical [5] to the simultaneous correlation functions of Eulerian velocity differences (1), i.e. $\boldsymbol{\mathcal{F}}^{(p)}_{n,1}(0) = \boldsymbol{F}_n$. Integrate (14) in the interval $(-\infty, 0)$, use the evaluation (15), and derive

$$R\boldsymbol{F}_n \sim \tau^{(p+1)}_{n+1,1}\boldsymbol{F}_{n+1} \ . \quad (17)$$

We see that from the point of view of scaling there is no p dependence in this equation: for different values of p only the coefficients can change. We thus estimate

$$\tau_{n,1}(R) \sim RS_{n-1}(R)/S_n(R) \propto R^{z_{n,1}} \ . \quad (18)$$

Here we introduced the dynamical scaling exponent $z_{n,1}$ that characterizes this time and found that

$$z_{n,1} = 1 + \zeta_{n-1} - \zeta_n \ . \quad (19)$$

Consider next the three-time quantity that is obtained from $\boldsymbol{\mathcal{F}}_n$ by choosing $t_i = t + \tau_1$ for $i \le p$, $t_i = t + \tau_2$ for $p < i \le p + q$, and $t_i = t$ for $i > p + q$. We denote this quantity as $\boldsymbol{\mathcal{F}}^{(p,q)}_{n,2}(\tau_1, \tau_2)$, omitting again the rest of the arguments. We define the decorrelation time $\tau^{(p,q)}_{n,2}$ of this quantity by

$$\int_{-\infty}^{0} d\tau_1 d\tau_2 \boldsymbol{\mathcal{F}}^{(p,q)}_{n,2}(\tau_1, \tau_2) \equiv [\tau^{(p,q)}_{n,2}]^2 \boldsymbol{\mathcal{F}}^{(p,q)}_{n,2}(0, 0) \ . \quad (20)$$

One could think naively that the decorrelation time $\tau^{(p,q)}_{n,2}$ is of the same order as (18). The calculation leads to a different result. To see this calculate the double derivative of $\boldsymbol{\mathcal{F}}^{(p,q)}_{n,2}(\tau_1, \tau_2)$ with respect to τ_1 and τ_2. This results in a new balance equation. For the fully unfused situation, and in the limit $\nu \to 0$ we find

$$\partial^2 \boldsymbol{\mathcal{F}}^{(p,q)}_{n,2}(\tau_1, \tau_2)/\partial\tau_1\partial\tau_2 + \boldsymbol{\mathcal{D}}^{(p,q)}_{n,2}(\tau_1, \tau_2) = 0 \ , \quad (21)$$

where now

$$\boldsymbol{\mathcal{D}}^{(p,q)}_{n,2}(\tau_1, \tau_2) = \sum_{j=1}^{p} \sum_{k=p+1}^{p+q} \langle \boldsymbol{\mathcal{W}}_1 \dots \hat{\boldsymbol{\mathcal{L}}}_j \boldsymbol{\mathcal{W}}_j \dots \hat{\boldsymbol{\mathcal{L}}}_k \boldsymbol{\mathcal{W}}_k \dots \boldsymbol{\mathcal{W}}_n \rangle .$$

In the RHS of (21) we neglected two terms that vanish in the limit $\nu \to 0$. The expression for $\boldsymbol{\mathcal{D}}^{(p,q)}_{n,2}(\tau_1, \tau_2)$ contains two space-integrals that originate from the two projection operators which are hidden in $\hat{\boldsymbol{\mathcal{L}}}_j$ and $\hat{\boldsymbol{\mathcal{L}}}_k$. Using the same ideas when all the separations are of the order of R we can estimate with impunity

$$\boldsymbol{\mathcal{D}}^{(p,q)}_{n,2}(\tau_1, \tau_2) \sim \boldsymbol{\mathcal{F}}^{(p+1,q+1)}_{n+2,2}(\tau_1, \tau_2)/R^2 \ . \quad (22)$$

Integrating now Eq.(21) over τ_1 and τ_2 in the interval $[-\infty, 0]$ and remembering that $\boldsymbol{\mathcal{F}}^{(p,q)}_{n,2}(0, 0) = \boldsymbol{F}_n$ we find

$$R\boldsymbol{F}_n \sim \boldsymbol{F}_{n+2}[\tau^{(p+1,q+1)}_{n+2,2}]^2 \ . \quad (23)$$

As before the scaling exponents are independent of p and q and we we can introduce a notation $\tau_{n,2} \sim \tau^{(p,q)}_{n,2}$. Up to p, q-dependent coefficients

$$[\tau_{n,2}(R)]^2 \sim R^2 S_{n-2}(R)/S_n(R) \propto [R^{z_{n,2}}]^2 \ . \quad (24)$$

We see that the naive expectation is not realized. The scaling exponent of the present time is different from (19):

$$z_{n,2} = 1 + (\zeta_{n-2} - \zeta_n)/2 \ . \quad (25)$$

The difference between the two scaling exponents $z_{n,1} - z_{n,2} = \zeta_{n-1} - (\zeta_n + \zeta_{n-2})/2$. This difference is zero for linear scaling, meaning that in that case the naive expectation that the time scales are identical is correct. On the other hand for the situation of multiscaling the Hoelder inequalities require the difference to be positive. Accordingly, for $R \ll L$ we have $\tau_{n,2}(R) \gg \tau_{n,1}(R)$.

We can proceed with correlation functions that depend on m time differences. Omitting the upper indices which are irrelevant for the scaling exponents we denote the correlation function as $\boldsymbol{\mathcal{F}}_{n,m}(\tau_1 \dots \tau_m)$, and establish the exact scaling law for its decorrelation time. The definition of the decorrelation time is

$$\int_{-\infty}^{0} d\tau_1 \dots d\tau_m \boldsymbol{\mathcal{F}}_{n,m}(\tau_1 \dots \tau_m) \equiv [\tau_{n,m}]^m \boldsymbol{\mathcal{F}}_{n,m}(0 \dots 0). \quad (26)$$

Repeating the steps described above we find the dynamical scaling exponent that characterizes $\tau_{n,m}$ when all the separations are of the order of R, $\tau_{n.m} \propto R^{z_{n,m}}$:

$$z_{n,m} = 1 + (\zeta_{n-m} - \zeta_n)/m \ , \quad n - m \le 2. \quad (27)$$

One can see, using the Hoelder inequalities, that $z_{n,m}$ is a nonincreasing function of m for fixed n, and in a multiscaling situation they are decreasing. The meaning is that the larger m is the *longer* is the decorrelation time of the corresponding $m+1$-time correlation function, $\tau_{n,p}(R) \gg \tau_{n,q}(R)$ for $p < q$.

To gain further understanding of the properties of the time correlation functions, we consider higher order temporal moments of the two-time correlation functions:

$$\int_{-\infty}^{0} d\tau \tau^{k-1} \boldsymbol{\mathcal{F}}^{(p)}_{n,1}(\tau) \equiv (\overline{\tau^k})^{(p)}_{n,1} \boldsymbol{\mathcal{F}}^{(p)}_{n,1}(0) \ . \quad (28)$$

The intuitive meaning of $(\overline{\tau^k})^{(p)}_{n,1}$ is a *k-order decorrelation moment* of $\boldsymbol{\mathcal{F}}^{(p)}_{n,1}(R, \tau)$, and note that its dimension is (time)^k. The first order decorrelation moment is the previously defined decorrelation time $\tau^{(p)}_{n,1}$. To find the scaling exponents of these quantities we start with Eq.(14), multiply by τ^k, and integrate over τ in the interval $(-\infty, 0)$. Using the evaluation (15) and assuming convergence of the integrals over τ we derive

$$-k \int_{-\infty}^{0} \boldsymbol{\mathcal{F}}^{(p)}_{n,1} \tau^{k-1} d\tau \sim \frac{1}{R} \int_{-\infty}^{0} \boldsymbol{\mathcal{F}}^{(p+1)}_{n+1,1} \tau^k d\tau \ , \quad (29)$$

where we have integrated by parts on the LHS. We stress that in deriving this equation we assert that $k + 1$ moments exist; this is not known apriori. Using the definition (28) and for all the separations of the order of R we find the recurrence relation

$$RS_n(R)(\overline{\tau^k})^{(p)}_{n,1} \sim S_{n+1}(R)(\overline{\tau^k})^{(p+1)}_{n+1,1} \ . \quad (30)$$

The solution is

$$[\overline{\tau^k}]^{(p)}_{n,1} \sim (\tau_{n,k})^k \sim \frac{R^k S_{n-k}(R)}{S_n(R)} \ . \tag{31}$$

for $k \le n-2$. The procedure does not yield information about higher k values.

We learn from the analysis of the moments that there is no single typical time which characterizes the τ dependence of $\boldsymbol{\mathcal{F}}^{(p)}_{n,1}$. There is no simple "dynamical scaling exponent" z that can be used to collapse the time dependence in the form $\boldsymbol{\mathcal{F}}^{(p)}_{n,1}(\tau) \sim R^{\zeta_n} f(\tau/R^z)$. Even the two-time correlation function is not a scale invariant object. In this respect it is similar to the probability distribution function of the velocity differences across a scale R, for which the spectrum of ζ_n is a reflection of the lack of scale invariance.

The main conclusion of this Letter is that all the dynamical scaling exponents can be determined from the knowledge of the scaling exponents ζ_n of the standard structure functions $S_n(R)$. All the scaling relations that were obtained above can be easily remembered using the following simple rule.

To get the dynamical scaling exponent every integral over τ in the definition of the decorrelation time (26), and every factor τ in the definition of the moments (28) can be traded for a factor of $R/\mathcal{W}$ *within the average* of the correlation function involved. The dynamical exponent is determined by the resulting scaling exponents of the resulting simultaneous correlation function.

The deep reason for this simple rule is the non-perturbative locality (convergence) of the integrals appearing in $\mathcal{D}_n$. Because of this locality one can estimate from the equation of motion (6) $1/\tau \sim \partial/\partial\tau \sim \boldsymbol{\mathcal{W}} \cdot \boldsymbol{\nabla} \sim \mathcal{W}(R)/R$. This means that we can use the substitutions

$$\tau, \int d\tau \Rightarrow R/\mathcal{W}(R) \tag{32}$$

as long as we use them within the average, and when all the separations are of the order of R. We propose to refer to this substitution rule as "weak dynamical similarity", where "weak" stands for a reminder that the rules can be used *only* under the averaging procedure, and *only* for scaling purposes. The same property of locality of the interaction integrals was shown [10,11] to yield another set of bridge relations between scaling exponents of correlation functions of gradient fields and the scaling exponents ζ_n. Those relations can be summarized by another substitution rule that we refer to as "weak dissipative similarity"; It follows from equating the viscous and nonlinear terms in the equations of motion:

$$\nu\nabla^2 \Rightarrow \mathcal{W}(R)/R. \tag{33}$$

Again "weak" refers to the reminder that we are only allowed to use these substitutions for scaling purposes within the average. Note that our weak dissipative similarity rule is weaker than the Kolmogorov refined similarity hypothesis which states the *dynamical* relationship $(\nabla\mathcal{W}(R))^2 \sim \mathcal{W}^3/R$. Both our rules are derived from first principles, while Kolmogorov's hypothesis is a guess.

Finally, we should ask whether the results presented above are particular to the time correlation function of BL-velocity differences, or do they reflect intrinsic scaling properties that are shared by other dynamical presentations like the standard Lagrangian velocity fields. The answer is that the results are general; all that we have used are the property of convergence of the interaction integrals, and the fact that the simultaneous correlation functions of the BL-fields are the same as those of the Eulerian velocities. These properties hold also for Lagrangian velocities, and in fact for any sensible choice of velocity representation in which the sweeping effect is eliminated. Accordingly we state that the dynamical exponent are invariant to the representation and in particular will be the same for many-time correlation functions of Lagrangian velocity differences.

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